

Extensions of the I-MMSE Relation

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Abstract

Unveiling a fundamental link between information theory and estimation theory, the I-MMSE relation by Guo, Shamai and Verdú [4] has great theoretical significance and numerous practical applications. On the other hand, its influences to date have been restricted to channels without feedback and memory, due to the lack of extensions of the I-MMSE relation to such channels. In this paper, we propose extensions of the I-MMSE relation for discrete and continuous-time Gaussian channels with feedback or memory. Our approach is based on a very simple observation, which can be applied to other scenarios, such as a simple and direct proof of the classical de Bruijn's identity.

1 Introduction

Consider the following discrete-time memoryless Gaussian channel

$$Y = \sqrt{snr}X + Z, \quad (1.1)$$

where snr denotes the signal-to-noise ratio of the channel, X and Y denote the input and output of the channel, respectively, and the standard normally distributed noise Z is independent of X . An interesting recent result by Guo, Shamai and Verdú [4] states that for any channel input X with $E[X^2] < \infty$,

$$\frac{d}{dsnr}I(X; Y) = \frac{1}{2}E[(X - E[X|Y])^2], \quad (1.2)$$

where the left hand side is the derivative of $I(X; Y)$ with respect to snr , and the right-hand side is half the so-called minimum mean-square error (MMSE), which corresponds to the best estimation of X given the observation Y . The I-MMSE relation as in (1.2) carries over verbatim to linear vector Gaussian channels and has been widely extended to continuous-time Gaussian channels [4], abstract Gaussian channels [17], additive channels [5], arbitrary channels [10], derivatives with respect to arbitrary parameterizations [9], higher order derivatives [11], and so on.

Unveiling an important link between information theory and estimation theory, the I-MMSE relation as above and its numerous extensions are of fundamental significance to relevant areas in these two fields and have been exerting far-reaching influences over a wide-range of topics. Representative applications include, but not limited to, power allocation of parallel Gaussian channels [8], analysis of extrinsic information of code ensembles [12], Gaussian broadcast channels [6], Gaussian wiretap channels [6, 1], Gaussian interference channels [2], interference alignment [16], a simple proof of the classical entropy power inequality [15]. For a comprehensive reference to the applications of the I-MMSE relation and its extensions, we refer to [13] .

On the other hand, all the applications of the I-MMSE relation to date haven been restricted to channels without feedback and memory, due to the lack of extensions of the I-MMSE relation to such channels. In this regard, it is known that a “plain” generalization of the original I-MMSE relation to feedback channels should not be expected, which has been noted in [4], where an example is given to show that the exact I-MMSE relation fails to hold for some continuous-time feedback channel. In this paper, we remedy the situations with some explicit correctional terms (which vanish if the channel does not have feedback and memory) and extend the I-MMSE relation to channels with feedback or memory. Despite the fact that the I-MMSE relation have been examined from a number of perspectives (see its multiple proofs in [4]), our approach is still novel and powerful. As a matter of fact, other than recovering and extending the I-MMSE relation, our approach can be applied else where, such as yielding a simple and direct proof of the classical de Bruijn’s identity [14, 3]; see Section 2.2.

Our approach is based on a surprisingly simple idea, which can be roughly stated as follows: before taking derivative of an information-theoretical quantity with respect to certain parameters, we represent it as an expectation with respect to a probability space independent of the parameters. For illustrative purpose, in what follows, we consider the discrete-time Gaussian channel in (1.1) and review a “conventional” proof of (1.2) in [4] and compare it with ours.

First, note that for the channel in (1.1), taking derivative of $I(X; Y)$ is equivalent to that of $H(Y)$, which can be written as the expectation of $-\log f_Y(Y)$:

$$H(Y) = -E[\log f_Y(Y)].$$

In their fourth proof of (1.2), the authors of [4] choose the probability space, with respect to which the expectation as above is taken, to be the sample space of Y (with naturally induced measure), which obviously depends on snr . Under this probability space, $H(Y)$ is naturally expressed as:

$$H(Y) = - \int_{\mathbb{R}} f_Y(y) \log f_Y(y) dy.$$

Then, under some mild assumptions, the derivative of $H(Y)$ with respect to snr can penetrate into the integral, and then (1.2) follows from integration by parts and other straightforward computations.

Under our approach, we would rather choose a probability space independent of snr . For example, choosing the probability space to be the sample space of (X, Z) , we will express

$H(Y)$ as

$$H(Y) = - \int_{\mathbb{R}} \int_{\mathbb{R}} f_X(x) f_Z(z) \log f_Y(\sqrt{snr}x + z) dx dz.$$

It turns out such a seemingly innocent shift of viewpoint will render the follow-up computations rather simple and direct before reaching (1.2); and most importantly, when applied to channels with feedback or memory, it naturally leads to extensions of the I-MMSE relation. For instance, consider the discrete-time Gaussian channel with feedback:

$$Y_i = \sqrt{snr} X_i(M, Y_1^{i-1}) + Z_i, \quad i = 1, 2, \dots, n$$

where the channel input X_i depends on the message M and the previous channel outputs Y_1^{i-1} . Using the above-mentioned approach, we will obtain the following extension (see Remark 3.1) of the I-MMSE relation:

$$\frac{d}{dsnr} I(X_1^n \rightarrow Y_1^n) = \frac{1}{2} \sum_{i=1}^n \mathbb{E} [(X_i - \mathbb{E}[X_i|Y_1^n])^2] + snr \sum_{i=1}^n \mathbb{E} \left[(X_i - \mathbb{E}[X_i|Y_1^n]) \frac{d}{dsnr} X_i \right], \quad (1.3)$$

where X_i is the abbreviated form of $X_i(M, Y_1^{i-1})$ and $I(X_1^n \rightarrow Y_1^n)$ is the directed information between X_1^n and Y_1^n . Directed information is a notion generalized from mutual information for feedback channels, and the second term in the right hand side of (1.3) is a correctional term, which vanishes when X_i does not depend on Y_1^{i-1} (*i.e.*, there is no feedback), so (1.3) is indeed an extension of the I-MMSE relation in (1.2) to discrete-time Gaussian channels with feedback. As elaborated later, the I-MMSE relation can also be extended to the continuous-time Gaussian channels with feedback or memory.

The remainder of the paper is organized as follows. In Section 2, based on the proposed approach, we give a new proof of the I-MMSE relation for discrete-time Gaussian channels, and a new proof of the classical de Bruijn's identity. We will present our extensions of the I-MMSE relation, the main results in this paper, in Section 3, which will be followed by an outlook for some promising future directions in Section 4.

2 New Proofs of Existing Results

In this section, to further illustrate the idea of our approach, we give new proofs of some existing results: the original I-MMSE relation in (1.2) and the classical de Bruijn's identity. To enhance the readability and emphasize the main idea, we omit some technical details, such as checking the conditions required for the interchange of differentiation and integration.

2.1 A new proof of the I-MMSE relation

In this section, we consider the Gaussian channel specified in (1.1) and give a new proof of (1.2). Here and throughout the paper, we replace \sqrt{snr} with ρ to avoid notational cumbersomeness during the computation; the derivative with respect to snr can be readily

obtained with an application of the chain rule. Then, under the new notation, we only have to prove that

$$\frac{d}{d\rho}I(X;Y) = \rho\mathbb{E}[(X - \mathbb{E}[X|Y])^2]. \quad (2.1)$$

Obviously, the conditional density of Y given $X = x$ by $f_{Y|X}(y|x) = \frac{1}{\sqrt{2\pi}}e^{(y-\rho x)^2/2}$, and the density function of Y can be computed as

$$f_Y(y) = \int_{\mathbb{R}} f_{Y|X}(y|x)f_X(x)dx.$$

It follows from the assumption that the channel is memoryless that

$$I(X;Y) = H(Y) - H(Y|X) = H(Y) - H(Z),$$

which, together with the fact that Z does not depend on ρ , implies that

$$\frac{d}{d\rho}I(X;Y) = -\frac{d}{d\rho}\mathbb{E}[\log f_Y(Y)] = -\mathbb{E}\left[\frac{1}{f_Y(Y)}\frac{d}{d\rho}f_Y(Y)\right].$$

Now, some straightforward computations yield

$$\begin{aligned} \frac{d}{d\rho}f_Y(Y) &= \frac{d}{d\rho} \int_{\mathbb{R}} f_{Y|X}(Y|x)f_X(x)dx \\ &= \int_{\mathbb{R}} (\rho X + Z - \rho x)X f_{Y|X}(Y|x)f_X(x)dx \\ &= -f_Y(Y) \int_{\mathbb{R}} (\rho X + Z - \rho x)X f_{X|Y}(x|Y)dx \end{aligned}$$

It then follows that

$$\begin{aligned} \frac{d}{d\rho}I(X;Y) &= \mathbb{E}\left[\int_{\mathbb{R}} (\rho X + Z - \rho x)X f_{X|Y}(x|Y)dx\right] \\ &= \mathbb{E}\left[\rho X^2 \int_{\mathbb{R}} f_{X|Y}(x|Y)dx\right] + \rho\mathbb{E}\left[XZ \int_{\mathbb{R}} f_{X|Y}(x|Y)dx\right] - \rho\mathbb{E}\left[X \int_{\mathbb{R}} x f_{X|Y}(x|Y)dx\right] \\ &= \rho\mathbb{E}[X^2] + 0 - \rho\mathbb{E}[X\mathbb{E}[X|Y]] \\ &= \rho\mathbb{E}[X^2 - \mathbb{E}^2[X|Y]] \\ &= \rho\mathbb{E}[(X - \mathbb{E}[X|Y])^2], \end{aligned}$$

as desired.

2.2 A new proof of de Bruijn's identity.

The following de Bruijn's identity is a fundamental relationship between the differential entropy and the Fisher information. Based on the proposed approach, we will give a new proof of this classical result.

Theorem 2.1. *Let X be any random variable with a finite variance and let Z be an independent standard normally distributed random variable. Then*

$$\frac{d}{dt}H(X + \sqrt{t}Z) = \frac{1}{2}J(X + \sqrt{t}Z), \quad (2.2)$$

where $J(\cdot)$ is the Fisher information.

Proof. First of all, define

$$Y = X + \sqrt{t}Z,$$

whose density function can be computed as

$$f_Y(y) = \int_{\mathbb{R}} f_X(x) f_{Y|X}(y|x) dx = \int_{\mathbb{R}} \frac{f_X(x)}{\sqrt{2\pi t}} e^{-(y-x)^2/(2t)} dx.$$

Immediately, we have

$$f_Y(Y) = f_Y(X + \sqrt{t}Z) = \int_{\mathbb{R}} \frac{f_X(x)}{\sqrt{2\pi t}} e^{-(X+\sqrt{t}Z-x)^2/(2t)} dx.$$

Now, taking the derivative, we obtain

$$\begin{aligned} \frac{d}{dt}f_Y(Y) &= \int_{\mathbb{R}} \frac{f_X(x)}{\sqrt{2\pi t}} e^{-(X+\sqrt{t}Z-x)^2/(2t)} \left(\frac{(X-x)(X+\sqrt{t}Z-x)}{2t^2} - \frac{1}{2t} \right) dx \\ &= \int_{\mathbb{R}} \left(\frac{(X-x)(X+\sqrt{t}Z-x)}{2t^2} - \frac{1}{2t} \right) f_{Y|X}(Y|x) f_X(x) dx \\ &= f_Y(Y) \int_{\mathbb{R}} \left(\frac{(X-x)(Y-x)}{2t^2} + \frac{1}{2t} \right) f_{X|Y}(x|Y) dx. \end{aligned}$$

It then follows that

$$\begin{aligned} \frac{d}{dt}H(Y) &= -\frac{d}{dt}\mathbb{E}[\log f_Y(Y)] = -\mathbb{E}\left[\frac{1}{f_Y(Y)} \frac{d}{dt}f_Y(Y)\right] \\ &= \mathbb{E}\left[\int_{\mathbb{R}} \left(-\frac{(X-x)(Y-x)}{2t^2} + \frac{1}{2t} \right) f_{X|Y}(x|Y) dx\right] \\ &= \frac{\mathbb{E}[-XY + (X+Y)\mathbb{E}[X|Y] - \mathbb{E}[X^2|Y]]}{2t^2} + \frac{1}{2t} \\ &= \frac{-\mathbb{E}[X^2] + \mathbb{E}[\mathbb{E}^2[X|Y]]}{2t^2} + \frac{1}{2t}. \end{aligned} \quad (2.3)$$

On the other hand, similarly as above,

$$f'_Y(Y) = \int_{\mathbb{R}} \frac{f_X(x)}{\sqrt{2\pi t}} e^{-(Y-x)^2/(2t)} \frac{x-Y}{t} dx = f_Y(Y) \int_{\mathbb{R}} \frac{x-Y}{t} f_{X|Y}(x|Y) dx,$$

It then follows that the right hand side of (2.2) can be computed as

$$\begin{aligned} J(Y) &= \mathbb{E} \left[\left(\frac{f'_Y(Y)}{f_Y(Y)} \right)^2 \right] \\ &= \frac{\mathbb{E}[\mathbb{E}^2[X|Y] + Y^2 - 2\mathbb{E}[X|Y]Y]}{t^2} \\ &= \frac{\mathbb{E}[\mathbb{E}^2[X|Y]] + \mathbb{E}[Y^2] - 2\mathbb{E}[XY]}{t^2}, \end{aligned}$$

which, by the fact that $t = \mathbb{E}[(X - Y)^2]$, is equal to (2.3), the left hand side of (2.2). The theorem then immediately follows. \square

3 Main Results

In this section, using the ideas and techniques illustrated in Section 2, we give extensions of the I-MMSE relations to channels with feedback or output memory.

3.1 Extensions to discrete-time channels

We start with the following general theorem on a discrete-time system:

Theorem 3.2. *Consider the following discrete-time system*

$$Y_i = \rho g_i(W_i, Y_1^{i-1}) + Z_i, \quad i = 1, \dots, n, \quad (3.4)$$

where all W_i are independent of all Z_i , which are i.i.d. standard normal random variables and $g_i(\cdot, \cdot)$ is a deterministic function differentiable in its second parameter. Then we have

$$\frac{d}{d\rho} I(W_1^n; Y_1^n) = \rho \sum_{i=1}^n \mathbb{E} [(g_i - \mathbb{E}[g_i|Y_1^n])^2] + \rho^2 \sum_{i=1}^n \mathbb{E} \left[(g_i - \mathbb{E}[g_i|Y_1^n]) \frac{d}{d\rho} g_i \right],$$

where we have written $g_i(W_i, Y_1^{i-1})$ simply as g_i .

Proof. Note that

$$I(W_1^n; Y_1^n) = H(Y_1^n) - \sum_{i=1}^n H(Y_i | W_1^n, Y_1^{i-1}) = H(Y_1^n) - nH(Z_1),$$

which immediately implies

$$\frac{d}{d\rho} I(W_1^n; Y_1^n) = -\mathbb{E} \left[\frac{d}{d\rho} \log f_{Y_1^n}(Y_1^n) \right] = -\mathbb{E} \left[\frac{1}{f_{Y_1^n}(Y_1^n)} \frac{d}{d\rho} f_{Y_1^n}(Y_1^n) \right].$$

Here, the above interchange between the expectation and differentiation needs verifications, which however are omitted due to the space limit.

In the remainder of the proof, we will omit the subscripts of the density functions. For instance, $f(y_1^n)$ means the density function of Y_1^n , $f(Y_1^n)$ means the density function of Y_1^n evaluated at Y_1^n , $f(y_1^n|w_1^n)$ means the conditional density function of Y_1^n given $W_1^n = w_1^n$.

Using the system assumption, we have

$$f(y_1^n|w_1^n) = \prod_{i=1}^n f(y_i|y_1^{i-1}, w_1^n) = \frac{1}{(\sqrt{2\pi})^n} \prod_{i=1}^n \exp\{-(y_i - \rho g_i(w_i, y_1^{i-1}))^2/2\},$$

and furthermore,

$$\begin{aligned} \frac{d}{d\rho} f(Y_1^n|w_1^n) &= \frac{1}{(\sqrt{2\pi})^n} \frac{d}{d\rho} \prod_{i=1}^n \exp\{-(Y_i - \rho g_i(w_i, Y_1^{i-1}))^2/2\} \\ &= \frac{1}{(\sqrt{2\pi})^n} \frac{d}{d\rho} \prod_{i=1}^n \exp\{-(\rho g_i(W_i, Y_1^{i-1}) - \rho g_i(w_i, Y_1^{i-1}) + Z_i)^2/2\} \\ &= -f(Y_1^n|w_1^n) \sum_{i=1}^n (Y_i - \rho g_i(w_i, Y_1^{i-1})) \left(g_i(W_i, Y_1^{i-1}) - g_i(w_i, Y_1^{i-1}) \right. \\ &\quad \left. + \rho \frac{d}{d\rho} (g_i(W_i, Y_1^{i-1}) - g_i(w_i, Y_1^{i-1})) \right). \end{aligned}$$

It then follows that

$$\begin{aligned} \frac{d}{d\rho} f(Y_1^n) &= \frac{d}{d\rho} \int_{\mathbb{R}^n} f(Y_1^n|w_1^n) f(w_1^n) dw_1^n \\ &= \int_{\mathbb{R}^n} \frac{d}{d\rho} f(Y_1^n|w_1^n) f(w_1^n) dw_1^n \\ &= - \int_{\mathbb{R}^n} \sum_{i=1}^n (Y_i - \rho g_i(w_i, Y_1^{i-1})) \left(g_i(W_i, Y_1^{i-1}) - g_i(w_i, Y_1^{i-1}) \right. \\ &\quad \left. + \rho \frac{d}{d\rho} (g_i(W_i, Y_1^{i-1}) - g_i(w_i, Y_1^{i-1})) \right) f(Y_1^n|w_1^n) f(w_1^n) dw_1^n \\ &= -f(Y_1^n) \int_{\mathbb{R}^n} \sum_{i=1}^n (Y_i - \rho g_i(w_i, Y_1^{i-1})) \left(g_i(W_i, Y_1^{i-1}) - g_i(w_i, Y_1^{i-1}) \right. \\ &\quad \left. + \rho \frac{d}{d\rho} (g_i(W_i, Y_1^{i-1}) - g_i(w_i, Y_1^{i-1})) \right) f(w_1^n|Y_1^n) dw_1^n. \end{aligned}$$

Writing $g_i(W_i, Y_1^{i-1})$, $g_i(w_i, Y_1^{i-1})$ as g_i , \tilde{g}_i , respectively, and using the fact that for any function φ ,

$$\int_{\mathbb{R}^n} \varphi(w_1^n, Y_1^n) f(w_1^n|Y_1^n) dw_1^n = \mathbb{E}[\varphi(W_1^n, Y_1^n)|Y_1^n],$$

we further compute

$$\begin{aligned} \frac{d}{d\rho} f(Y_1^n) &= -f(Y_1^n) \sum_{i=1}^n \int_{\mathbb{R}^n} (Y_i - \rho \tilde{g}_i) \left((g_i + \rho \frac{d}{d\rho} g_i) - (\tilde{g}_i + \rho \frac{d}{d\rho} \tilde{g}_i) \right) f(w_1^n|Y_1^n) dw_1^n \\ &= -f(Y_1^n) \sum_{i=1}^n \left((g_i + \rho \frac{d}{d\rho} g_i) \mathbb{E}[(Y_i - \rho g_i)|Y_1^n] - \mathbb{E} \left[(g_i + \rho \frac{d}{d\rho} g_i)(Y_i - \rho g_i) \middle| Y_1^n \right] \right). \end{aligned}$$

Similarly continue as in the proof of (2.1), we eventually obtain

$$\begin{aligned}\frac{d}{d\rho}I(W_1^n; Y_1^n) &= \sum_{i=1}^n \left(\mathbb{E} \left[(g_i + \rho \frac{d}{d\rho} g_i)(Y_i - \rho \mathbb{E}[g_i | Y_1^n]) \right] - \mathbb{E} \left[(g_i + \rho \frac{d}{d\rho} g_i)(Y_i - \rho g_i) \right] \right) \\ &= \rho \sum_{i=1}^n \mathbb{E} [(g_i - \mathbb{E}(g_i | Y_1^n))^2] + \rho^2 \sum_{i=1}^n \mathbb{E} \left[(g_i - \mathbb{E}(g_i | Y_1^n)) \frac{d}{d\rho} g_i \right],\end{aligned}$$

as desired. \square

Remark 3.1. Consider the discrete-time system as in (3.4). Rewriting all W_i as M and each g_i as X_i , we then have the following discrete-time Gaussian channel with feedback:

$$Y_i = \sqrt{snr} X_i(M, Y_1^{i-1}) + Z_i, \quad i = 1, 2, \dots, n$$

where M is interpreted as the message be transmitted and X_i, Y_i are the channel inputs, outputs, respectively. It is well known that for such a feedback channel,

$$I(X_1^n \rightarrow Y_1^n) = I(M; Y_1^n),$$

where $I(X_1^n \rightarrow Y_1^n)$ is the directed information between X_1^n and Y_1^n . Then, applying Theorem 3.2 and the chain rule for taking derivative, we have

$$\frac{d}{dsnr} I(X_1^n \rightarrow Y_1^n) = \frac{1}{2} \sum_{i=1}^n \mathbb{E} [(X_i - \mathbb{E}[X_i | Y_1^n])^2] + snr \sum_{i=1}^n \mathbb{E} \left[(X_i - \mathbb{E}[X_i | Y_1^n]) \frac{d}{dsnr} X_i \right],$$

where $X_i = X_i(M, Y_1^{i-1})$. This yields an extension of the I-MMSE relation to discrete-time Gaussian channels with feedback.

Remark 3.2. Alternatively, rewriting each W_i as X_i , we will have the following discrete-time Gaussian channel with output memory:

$$Y_i = \sqrt{snr} g_i(X_i, Y_1^{i-1}) + Z_i, \quad i = 1, 2, \dots, n$$

where g_i is interpreted as “part” of the channel and X_i, Y_i are the channel inputs, outputs, respectively. Then, by Theorem 3.2 and the chain rule, we obtain

$$\frac{d}{dsnr} I(X_1^n; Y_1^n) = \frac{1}{2} \sum_{i=1}^n \mathbb{E} [g_i - \mathbb{E}[g_i | Y_1^n]]^2 + snr \sum_{i=1}^n \mathbb{E} \left[(g_i - \mathbb{E}[g_i | Y_1^n]) \frac{d}{dsnr} g_i \right],$$

where $g_i = g_i(X_i, Y_1^{i-1})$. This yields an extension of the I-MMSE relation to discrete-time Gaussian channels with output memory.

3.2 Extensions to continuous-time channels

We start with a general theorem on a continuous-time system:

Theorem 3.3. *Consider the following continuous-time system*

$$Y(t) = \rho \int_0^t g(s, W(s), Y_0^s) ds + B(t), \quad t \in [0, T], \quad (3.5)$$

where $W(t)$ is independent of $B(t)$, which is the standard Brownian motion, and $g(\cdot, \cdot, \cdot)$ is a deterministic function differentiable in the third parameter (in the sense of Fréchet). We then have

$$\frac{d}{d\rho} I(W_0^T; Y_0^T) = \rho \int_0^T \mathbb{E}[(g(s) - \mathbb{E}[g(s)|Y_0^T])^2] ds + \rho^2 \int_0^T \mathbb{E} \left[(g(s) - \mathbb{E}[g(s)|Y_0^T]) \frac{d}{d\rho} g(s) \right] ds,$$

where we have written $g(s, X(s), Y_0^{s-})$ simply as $g(s)$.

Proof. Fix $W = w$ and let \tilde{Y} be such that

$$\tilde{Y}(t) = \int_0^t g(s, w(s), \tilde{Y}(s)) ds + B(t), \quad t \in [0, T].$$

Then, by Girsanov's theorem (see, e.g., Theorem 7.1 in [7]), we have

$$\frac{d\mu_{Y|W}}{d\mu_B}(\tilde{Y}|w) = \exp \left\{ \rho \int_0^T g(s, w(s), \tilde{Y}_0^s) d\tilde{Y}(s) - \frac{\rho^2}{2} \int_0^T g(s, w(s), \tilde{Y}_0^s) ds \right\}.$$

It then follows from the fact that $\mu_{Y_w} \sim \mu_B \sim \mu_Y$ (" \sim " means equivalent) and Lemma 4.10 in [7] that

$$\frac{d\mu_{Y|W}}{d\mu_B}(Y|w) = \exp \left\{ \rho \int_0^T g(s, w(s), Y_0^s) dY(s) - \frac{\rho^2}{2} \int_0^T g(s, w(s), Y_0^s) ds \right\}.$$

It then follows that

$$\begin{aligned} I(W_0^T; Y_0^T) &= \mathbb{E} \left[\log \frac{d\mu_{WY}}{d(\mu_W \times \mu_Y)}(W_0^T, Y_0^T) \right] \\ &= \mathbb{E} \left[\log \frac{d\mu_{Y|W}}{d\mu_B}(Y_0^T|W_0^T) \right] - \mathbb{E} \left[\log \frac{d\mu_Y}{d\mu_B}(Y_0^T) \right] \\ &= \frac{\rho^2}{2} \int_0^T \mathbb{E}[g^2(s)] ds - \mathbb{E} \left[\log \frac{d\mu_Y}{d\mu_B}(Y_0^T) \right]. \end{aligned}$$

Taking derivative with respect to ρ , we then have

$$\begin{aligned} \frac{d}{d\rho} I(W_0^T; Y_0^T) &= \rho \int_0^T \mathbb{E}[g^2(s)] ds + \frac{\rho^2}{2} \frac{d}{d\rho} \int_0^T \mathbb{E}[g^2(s)] ds - \frac{d}{d\rho} \mathbb{E} \left[\log \frac{d\mu_Y}{d\mu_B}(Y_0^T) \right] \\ &= \rho \int_0^T \mathbb{E}[g^2(s)] ds + \rho^2 \int_0^T \mathbb{E} \left[g(s) \frac{d}{d\rho} g(s) \right] ds - \frac{d}{d\rho} \mathbb{E} \left[\log \frac{d\mu_Y}{d\mu_B}(Y_0^T) \right]. \end{aligned}$$

Here and hereafter, the interchanges between the expectation and differentiation need verifications, which however are omitted due to the space limit.

Writing $g(s, w(s), Y_0^s)$ as $\tilde{g}(s)$, we have

$$\begin{aligned}
\frac{d}{d\rho} \left(\frac{d\mu_Y}{d\mu_B}(Y_0^T) \right) &= \frac{d}{d\rho} \int \frac{d\mu_{Y|W}}{d\mu_B}(Y_0^T|w) \mu_W(dw) \\
&= \frac{d}{d\rho} \int \exp \left\{ \rho \int_0^T \tilde{g}(s) dY(s) - \frac{\rho^2}{2} \int_0^T \tilde{g}^2(s) ds \right\} \mu_W(dw) \\
&= \frac{d}{d\rho} \int \exp \left\{ \rho^2 \int_0^T \tilde{g}(s) g(s) ds + \rho \int_0^T \tilde{g}(s) dB(s) - \frac{\rho^2}{2} \int_0^T \tilde{g}^2(s) ds \right\} \mu_W(dw) \\
&= \int \left(\int_0^T \tilde{g}(s) dY(s) + \rho \int_0^T \frac{d}{d\rho} \tilde{g}(s) dY(s) + \rho \int_0^T \tilde{g}(s) (g(s) - \tilde{g}(s)) ds \right. \\
&\quad \left. + \rho^2 \int_0^T \tilde{g}(s) \frac{d}{d\rho} (g(s) - \tilde{g}(s)) ds \right) \frac{d\mu_{WY}}{d\mu_B}(dw, Y) \\
&= \frac{d\mu_Y}{d\mu_B}(Y) \int \left(\int_0^T \tilde{g}(s) dY(s) + \rho \int_0^T \frac{d}{d\rho} \tilde{g}(s) dY(s) + \rho \int_0^T \tilde{g}(s) (g(s) - \tilde{g}(s)) ds \right. \\
&\quad \left. + \rho^2 \int_0^T \tilde{g}(s) \frac{d}{d\rho} (g(s) - \tilde{g}(s)) ds \right) \mu_{W|Y}(dw|Y) \\
&= \frac{d\mu_Y}{d\mu_B}(Y) \left(\int_0^T \mathbb{E}[g(s)|Y_0^T] dY(s) + \rho \int_0^T \frac{d}{d\rho} \mathbb{E}[g(s)|Y_0^T] dY(s) \right. \\
&\quad \left. + \rho \int_0^T (\mathbb{E}[g(s)|Y_0^T] g(s) - \mathbb{E}[g^2(s)|Y_0^T]) ds + \rho^2 \int_0^T \left(\frac{d}{d\rho} g(s) \mathbb{E}[g(s)] - \mathbb{E} \left[g(s) \frac{d}{d\rho} g(s) \middle| Y_0^T \right] \right) ds \right)
\end{aligned}$$

Note that by the properties of conditional expectation and Itô integral, we have

$$\mathbb{E} \left[\int_0^T \mathbb{E}[g(s)|Y_0^T] dY(s) \right] = \mathbb{E} \left[\mathbb{E} \left[\int_0^T g(s) dY(s) \middle| Y_0^T \right] \right] = \mathbb{E} \left[\int_0^T g(s) dY(s) \right] = \rho \int_0^t \mathbb{E}[g^2(s)] ds,$$

and similarly,

$$\mathbb{E} \left[\int_0^T \mathbb{E}[g^2(s)|Y_0^T] ds \right] = \int_0^T \mathbb{E}[g^2(s)] ds,$$

and

$$\rho \mathbb{E} \left[\int_0^T \frac{d}{d\rho} \mathbb{E}[g(s)|Y_0^T] dY(s) \right] = \rho^2 \int_0^T \mathbb{E} \left[g(s) \frac{d}{d\rho} g(s) \right] ds = \rho^2 \mathbb{E} \left[\int_0^T \mathbb{E} \left[g(s) \frac{d}{d\rho} g(s) \right] ds \right].$$

It then follows that

$$\begin{aligned}
\mathbb{E} \left[\frac{d}{d\rho} \left(\frac{d\mu_Y}{d\mu_B}(Y_0^T) \right) / \frac{d\mu_Y}{d\mu_B}(Y_0^T) \right] &= \mathbb{E} \left[\int_0^T \mathbb{E}[g(s)|Y_0^T] dY(s) + \rho \int_0^T \frac{d}{d\rho} \mathbb{E}[g(s)|Y_0^T] dY(s) \right. \\
&\quad \left. + \rho \int_0^T (\mathbb{E}[g(s)|Y_0^T] g(s) - \mathbb{E}[g^2(s)|Y_0^T]) ds + \rho^2 \int_0^T \left(\mathbb{E}[g(s)|Y_0^T] \frac{d}{d\rho} g(s) - \mathbb{E} \left[g(s) \frac{d}{d\rho} g(s) \middle| Y_0^T \right] \right) ds \right] \\
&= \rho \int_0^T \mathbb{E}[\mathbb{E}[g(s)|Y_0^T] g(s)] ds + \rho^2 \int_0^T \mathbb{E} \left[\mathbb{E}[g(s)|Y_0^T] \frac{d}{d\rho} g(s) \right] ds
\end{aligned}$$

So we have

$$\begin{aligned}
\frac{d}{d\rho}I(W_0^T; Y_0^T) &= \rho \int_0^T \mathbb{E}[g^2(s)]ds + \rho^2 \int_0^T \mathbb{E} \left[g(s) \frac{d}{d\rho} g(s) \right] ds \\
&\quad - \rho \int_0^T \mathbb{E}[\mathbb{E}[g(s)|Y_0^T]g(s)]ds - \rho^2 \int_0^T \mathbb{E} \left[\mathbb{E}[g(s)|Y_0^T] \frac{d}{d\rho} g(s) \right] ds \\
&= \rho \int_0^T \mathbb{E}[(g(s) - \mathbb{E}[g(s)|Y_0^T])^2]ds + \rho^2 \int_0^T \mathbb{E} \left[(g(s) - \mathbb{E}[g(s)|Y_0^T]) \frac{d}{d\rho} g(s) \right] ds,
\end{aligned}$$

as desired. \square

Remark 3.3. *Parallel to Remarks 3.1, the continuous-time system in (3.5) can be interpreted as the following continuous-time Gaussian channel with feedback (below Y_0^s in (3.5) is replaced by Y_0^{s-} , which can be justified under very mild conditions by a continuity argument):*

$$Y(t) = \sqrt{snr} \int_0^t X(s, M, Y_0^{s-})ds + B(t), \quad t \in [0, T].$$

An application of Theorem 3.3 then yields

$$\frac{d}{dsnr}I(M; Y_0^T) = \frac{1}{2} \int_0^T \mathbb{E}[(X(s) - \mathbb{E}[X(s)|Y_0^T])^2]ds + snr \int_0^T \mathbb{E} \left[(X(s) - \mathbb{E}[X(s)|Y_0^T]) \frac{d}{dsnr} X(s) \right] ds,$$

where $X(s)$ is the abbreviated form of $X(s, M, Y^{s-})$. This gives an extension of the I-MMSE relation to continuous-time Gaussian channels with feedback.

Parallel to Remarks 3.2, it can be also interpreted as the following continuous-time Gaussian channel with output memory:

$$Y(t) = \sqrt{snr} \int_0^t g(s, X(s), Y_0^{s-})ds + B(t), \quad t \in [0, T].$$

An application of Theorem 3.3 then yields

$$\frac{d}{dsnr}I(X_0^T; Y_0^T) = \frac{1}{2} \int_0^T \mathbb{E}[(g(s) - \mathbb{E}[g(s)|Y_0^T])^2]ds + snr \int_0^T \mathbb{E} \left[(g(s) - \mathbb{E}[g(s)|Y_0^T]) \frac{d}{dsnr} g(s) \right] ds,$$

where $g(s)$ is the abbreviated form of $g(s, M, Y^{s-})$. This gives an extension of the I-MMSE relation to continuous-time Gaussian channels with output memory.

4 Conclusions and Future Work

Based on a simple yet powerful idea, we extend the well-known I-MMSE relation to channels with feedback or memory. Given the wide-range applications of the classical I-MMSE relation to various scenarios, one natural future direction is to examine the possible applications of the extensions to these scenarios when the feedback or memory are present. The new proof of the classical de Brunjin's identity also suggests possible applications of our approach to other scenarios.

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